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On the relation between $\Phi_{(1,2)}$ and $\Phi_{(1,5)}$ perturbed minimal models and unitarity.

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ABSTRACT

We consider the RSOS S -matrices of the $\Phi_{(1,5)}$ perturbed minimal models which have recently been found in the companion paper [1]. These S -matrices have some interesting properties, in particular, unitarity may be broken in a stronger sense than seen before, while one of the three classes of $\Phi_{(1,5)}$ perturbations (to be described) shares the same Thermodynamic Bethe Ansatz as a related $\Phi_{(1,2)}$ perturbation. We test these new S -matrices by the standard Truncated Conformal Space method, and further observe that in some cases the BA equations for two particle energy levels may be continued to complex rapidity to describe (a) single particle excitations and (b) complex eigenvalues of the Hamiltonian corresponding to non-unitary S -matrix elements. We make some comments on identities between characters in the two related models following from the fact that the two perturbed theories share the same breather sector.

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1 Introduction

This paper is the companion to [1], in which S -matrices were conjectured for $\Phi_{(1,5)}$ perturbations of Virasoro minimal models by RSOS reduction of the S -matrix of $a_2^{(2)}$ affine Toda theory. Throughout this paper we shall refer to equations in [1] as (1:nn).

The description of perturbed conformal field theories (PCFTs) as affine Toda theories [2] has provided much insight into their structure — the construction of the S -matrices for the particles in PCFTs relies upon the existence of an affine quantum symmetry of the Toda action [3] — but there are still many interesting aspects to investigate. In this paper we consider the $a_2^{(2)}$ affine Toda theory at imaginary coupling which can provide a model for both $\Phi_{(1,2)}$ and $\Phi_{(1,5)}$ perturbations of minimal models. In the simplest case, the $\Phi_{(1,2)}$ perturbation of the unitary Ising model (which has a minimal e_8 S -matrix) is described by the same Toda Lagrangian as the $\Phi_{(1,5)}$ perturbation of the non-unitary $\mathcal{M}_{3,16}$ model.

It is possible for the same Lagrangian to describe two such different models because the S -matrices of a PCFT are obtained by ‘RSOS restriction’ of the Toda S -matrices which relies upon a $\mathcal{U}_q(sl(2))$ subalgebra of the quantum symmetry algebra, and that one may be able to choose more than one such subalgebra leading to different S -matrices for the PCFTs. In this way it turns out that the same Lagrangian theory may indeed describe both a unitary and a non-unitary theory, and that even the particle spectra and ground-state thermodynamics of these two theories may be the same, yet the theories possess different S -matrices both being consistent solutions of the S -matrix bootstrap equations for the particle masses.

In the specific case of the $a_2^{(2)}$ Toda theory the RSOS restriction leading to the $\Phi_{(1,2)}$ PCFTs has been performed in [4, 5, 6] and that leading to the $\Phi_{(1,5)}$ perturbation in [1]. The way that the two perturbations which arise from them are related falls into one of three classes; the example considered by Martins in [7] is in the first class and we give the first examples of the other two classes in this paper, together with a unified description.

The $a_2^{(2)}$ affine Toda theory appears to describe a non-unitary theory, and while it is possible for the RSOS reduction of a non-unitary theory to yield a unitary scattering theory (as with the $\Phi_{(1,2)}$ perturbations of unitary minimal models) for most reductions the scattering theory is also non-unitary. However, for all the reductions proposed so far, this has been a rather weak form of non-unitarity, with the Hamiltonian having real eigenvalues bounded below, and simply the Hilbert space having an indefinite metric. However, in some of our models, we show how unitarity is violated in a stronger sense, and the Hamiltonian of the reduced theory may have complex eigenvalues as well.

The outline of the paper is as follows: in 2 we describe briefly the minimal models of conformal field theory and their perturbations and the way in which two different perturbed conformal minimal models, one perturbed by the $\Phi_{(1,2)}$ field and one by the $\Phi_{(1,5)}$ field, may be related to the same affine Toda Lagrangian field theory, and how the BRS reduction of the two different underlying Liouville theories naturally leads to two different S -matrices for the two perturbed minimal models.

We then pass to tests of the predicted S -matrices in the case of the $\Phi_{(1,5)}$ perturbations. In

section 3 we recall the predictions of the Thermodynamic Bethe Ansatz for the ground state energy of the particle system on a circle, which leads to predictions for the effective central charge, the mass-gaps and of the coefficient in the linear term in the asymptotic behaviour of the ground-state itself. We also show how in many cases the two perturbed minimal models which can be derived from Smirnov's S -matrix give the same TBA system, while in other cases this is not so. We also recall how the higher multi-particle energy levels can be predicted using the ordinary Bethe ansatz.

The first prediction can be checked analytically while the other three can be checked using the Truncated Conformal Space Approach (TCSA) of Yurov and Al. Zamolodchikov [9]. In section 4 we use this method to show that our predictions of the particle spectra and S -matrices are correct in three models, the minimal models $\mathcal{M}_{3,10}$, $\mathcal{M}_{3,14}$ and $\mathcal{M}_{3,16}$ perturbed by the field $\Phi_{(1,5)}$.

Furthermore we show that the Bethe Ansatz can be usefully continued to complex rapidities in two different cases. Firstly, we show that the ordinary Bethe Ansatz may also be a good description of a single particle state, when continued to imaginary rapidity. This presumably indicates that the Bethe Ansatz equations provide the dominant corrections to the full TBA equations for single particle states developed in [8]. Secondly, we show that for the non-unitary S -matrices the Bethe Ansatz still provides a good description of the complex eigenvalues analogous to two-particle states found in the TCSA investigation.

In section 5 we give some comments on the possible ways in which the S -matrix may be realised and the connection with the different partition functions and periodicity conditions which may occur for the perturbed models, and in section 6 we list some relations between the Virasoro characters of related minimal models. Finally we give our conclusions in section 7.

We give some details of the structure constants in the conformal field theories we use in appendix A.

2 Minimal models and their perturbations

A conformal field theory is characterised by the central charge c of the Virasoro algebra, the sets of conformal weights $\{h, \bar{h}\}$ of the primary fields $\Phi_a(z, \bar{z}) \sim \phi_h(z)\phi_{\bar{h}}(\bar{z})$ and the three point couplings of these fields $\langle \Phi_a | \Phi_b(1) | \Phi_c \rangle$. The minimal models of conformal field theory were first described in [10], and we denote them by $\mathcal{M}_{r,s}$ where r, s are coprime integers greater than 1. The Virasoro central charge in these models take the values

$$c = 1 - \frac{6(r-s)^2}{rs}, \quad (2.1)$$

and in $\mathcal{M}_{r,s}$ there are $(r-1)(s-1)/2$ possible values of h and \bar{h} , given by

$$h_{m,n} = \frac{(ms - nr)^2 - (r-s)^2}{4rs}, \quad 0 < m < r, \quad 0 < n < s, \quad (2.2)$$

although which actual pairs $\{h, \bar{h}\}$ are the conformal weights of fields is not determined a priori. For $|r-s|=1$ $\mathcal{M}_{r,s}$ is unitary, and the lowest value of h is $h_{1,1} = 0$; in all other cases

there is some allowed value of $h < 0$, but there is a unified formula for the effective central charge $c - 24h_{min}$,

$$c_{\text{eff}} \equiv c - 24h_{min} = 1 - \frac{6}{rs}. \quad (2.3)$$

One can consider perturbations of these conformally invariant field theories by the addition to the action of some term

$$\lambda \int d^2x \Phi_{(m,n)}(x), \quad (2.4)$$

where $\Phi_{(m,n)}$ is the conformal primary field corresponding to $h = \bar{h} = h_{m,n}$. First-order perturbation theory calculations, backed up by arguments based on counting of states, suggest that the perturbed field theory is integrable for (m,n) one of $(1,2)$, $(1,3)$, $(1,5)$, $(2,1)$, $(3,1)$ and $(5,1)$. For the unitary minimal models $\mathcal{M}_{r,r+1}$, only $(1,2)$, $(1,3)$ and $(2,1)$ are relevant perturbations, and consequently these have attracted the most attention to date.

Much insight can be gained into minimal models and perturbed minimal models by considering particular realisations as Lagrangian field theories. The Liouville action for a scalar field ϕ ,

$$S_0 = \int d^2z \left(\frac{1}{4\pi} \partial_z \phi \partial_{\bar{z}} \phi + \exp(\beta \sqrt{2} \phi) \right), \quad (2.5)$$

provides a realisation of the minimal models with $c = 13 + 6\beta^2 + 6\beta^{-2}$. The minimal models correspond to the purely imaginary values of $\beta = i\sqrt{r/s}$. Although this means that care must be taken with the interpretation of the action, Felder has provided a construction [11] which shows that the correlation functions defined in this theory are indeed those of the minimal models as found by Dotsenko and Fateev [12]: for the vacuum sector one must not consider the full Hilbert space of a scalar field but instead the reduced space $\text{Ker } Q / \text{Im} : Q^{r-1} :$ where Q is the screening charge

$$Q = \int dz \exp(\sqrt{2} \phi / \beta), \quad (2.6)$$

and $: Q^{r-1} :$ is an appropriately defined normal-ordered multiple integral. The vertex operators

$$\exp(-\alpha_{m,n} \phi / \sqrt{2}), \quad \alpha_{m,n} = (n-1)\beta - (m-1)/\beta, \quad (2.7)$$

transform as primary fields with conformal weight $h_{m,n}$. Consequently, one can construct a formal perturbed conformal field theory action as

$$S = S_0 + \lambda \int d^2z \exp(-\alpha_{m,n} \phi / \sqrt{2}). \quad (2.8)$$

For $(m,n) = (1,3)$ this gives the standard sine-Gordon action, whereas for $(m,n) = (1,2)$ or $(1,5)$ we obtain the Bullough-Dodd or Zhiber-Mikhailov-Shabat (ZMS) model, which is the affine Toda theory related to $a_2^{(2)}$.

It is this latter model in which we are most interested, and to fix on a normalisation of the coupling constant, we shall say that the ZMS model with coupling constant γ has the action

$$S_{ZMS}(\gamma) = \int d^2x \left((\partial_\mu \varphi)^2 + \frac{m^2}{\gamma} \left(\exp(i\sqrt{8\gamma}\varphi) + 2\exp(-i\sqrt{2\gamma}\varphi) \right) \right), \quad (2.9)$$

where m is a coupling constant with dimension of mass. The connection between the variables in (2.5) and (2.9) is given by

$$\phi = \sqrt{4\pi}\varphi, \quad \beta = i\sqrt{\frac{\gamma}{\pi}}. \quad (2.10)$$

It is immediately apparent that the ZMS action may be viewed as a perturbed conformal field theory in two different ways: each of the two exponentials may be thought of as the perturbation and the remainder as the Liouville action:

$$S_{ZMS}(\gamma) = \begin{cases} \mathcal{M}_{r,s} + \Phi_{(1,2)} & \gamma = \pi r/s \\ \mathcal{M}_{r',s'} + \Phi_{(1,5)} & \gamma = 4\pi r'/s' \end{cases}. \quad (2.11)$$

The actual values of r', s' depend on the arithmetic properties of r and s , since we require r', s' coprime. Hence the three distinct cases of table 1 can arise.

	$r \bmod 4$	r'	s'
I	0	$r/4$	s
II	2	$r/2$	$2s$
III	1, 3	r	$4s$

Table 1: Relation between (r, s) and (r', s')

These three cases differ in the way the effective central charges of $\mathcal{M}_{r,s}$ and $\mathcal{M}_{r',s'}$ are related. There are interesting examples of each of these cases as follows:

- I Here $c_{\text{eff}} = 1 - 6/rs$, $c_{\text{eff}}' = 1 - 24/(rs)$ so that $c_{\text{eff}}' < c_{\text{eff}}$. Since the effective central charge is a measure of the number of degrees of freedom in the UV theory, we expect that there will be fewer particles in $\mathcal{M}_{r',s'} + \Phi_{(1,5)}$ than in $\mathcal{M}_{r,s} + \Phi_{(1,2)}$.

The only $(1, 5)$ perturbation that has been investigated before falls into this class, being $\mathcal{M}_{2,9} + \Phi_{(1,5)}$ which is related to $\mathcal{M}_{8,9} + \Phi_{(1,2)}$ and which was discussed by Martins et al in [7, 6].

- II Here $c_{\text{eff}}' = c_{\text{eff}}$ and we expect the greatest similarity between the two models.

We will discuss two models, $\mathcal{M}_{6,5} + \Phi_{(1,2)}$ which is related to $\mathcal{M}_{3,10} + \Phi_{(1,5)}$, and $\mathcal{M}_{6,7} + \Phi_{(1,2)}$ which is related to $\mathcal{M}_{3,14} + \Phi_{(1,5)}$.

The first of these two models is unusual in that it can be expressed as a tensor product, $\mathcal{M}_{3,10} \equiv \mathcal{M}_{2,5} \otimes \mathcal{M}_{2,5}$. The fields in $\mathcal{M}_{3,10}$ can be classified into Z_2 -even and Z_2 -odd fields with respect to the Z_2 map provided by flipping the tensor product. The perturbing operator $\Phi_{(1,5)}$ is in the Z_2 -even sector and is nothing other than $(1 \otimes \Phi_{(1,2)} + \Phi_{(1,2)} \otimes 1)/\sqrt{2}$. As a result,

$$\mathcal{M}_{3,10} + \Phi_{(1,5)} = \left(\mathcal{M}_{2,5} + \Phi_{(1,2)} \right)^{\otimes 2}, \quad (2.12)$$

is identically true in the Z_2 even sector, and the spectrum consists of two self-conjugate particles of equal mass, each with the S -matrix of the $\mathcal{M}_{2,5} + \Phi_{(1,2)}$ model.

In the second case the spectrum consists of six particles $\{l, \bar{l}, L, h, \bar{h}, H\}$ for $\mathcal{M}_{6,7} + \Phi_{(1,2)}$ and $\{A, \bar{A}, C, B, \bar{B}, D\}$ for $\mathcal{M}_{3,14} + \Phi_{(1,5)}$. The S -matrix in the second case is unusual in that it is not possible to define S_{AB} and $S_{A\bar{B}}$ satisfying the usual unitarity condition, i.e. they are not pure phases. However, a formal application of the S -matrix-bootstrap gives $S_{AB} = S_1^{KL}, S_{A\bar{B}} = S_2^{KL}$ as in (1:79), and the remaining S -matrices can be found by application of the S -matrix-bootstrap to the poles in $S_{AB}, S_{A\bar{B}}$.

III In this case $c_{\text{eff}}' > c_{\text{eff}}$ and we expect there to be more particles in $\mathcal{M}_{r',s'} + \Phi_{(1,5)}$ than in $\mathcal{M}_{r,s} + \Phi_{(1,2)}$.

Here we consider the case of the thermal perturbation of the Ising model, i.e. $\mathcal{M}_{3,4} + \Phi_{(1,2)}$ for which the corresponding model is $\mathcal{M}_{3,16} + \Phi_{(1,5)}$.

Although the bootstrap was not completed in [1], again we expect that $\mathcal{M}_{3,16} + \Phi_{(1,5)}$ has the same peculiarities as $\mathcal{M}_{3,14} + \Phi_{(1,5)}$.

3 TBA predictions from the S -matrices.

As is well-known, the method of Thermodynamic Bethe Ansatz (TBA) is a powerful method for extracting the finite-size behaviour of a massive theory on a circle. One can also calculate this behaviour for the perturbed conformal field theory, both in perturbation theory about the conformal field theory and also numerically, and together these two methods provide a stringent test of the conjectured S -matrices.

We first formulate the conformal field theory on a cylinder of size R , and consider the Hamiltonian which propagates along the cylinder:

$$H = H_{CFT} + \lambda \int dx \Phi(z, \bar{z}) , \quad (3.1)$$

where

$$H_{CFT} = \frac{2\pi}{R} \left(L_0 + \bar{L}_0 - \frac{c}{12} \right) , \quad (3.2)$$

is the Hamiltonian of the ultraviolet CFT, $\Phi(z, \bar{z})$ is some relevant operator corresponding to the perturbation and λ is a dimensionful coupling. If the conformal dimension of the perturbing field $\Phi(z, \bar{z})$ is $d_\Phi = 2 - y$, then λ has mass dimension y . In terms of a specific mass unit m (which will be taken to be the mass of the lowest lying excitation of the model) we have

$$\lambda = \alpha m^y , \quad (3.3)$$

where α is a dimensionless constant. The perturbing Hamiltonian is given by the second term:

$$V = \lambda \int dx \Phi(z, \bar{z}) , \quad (3.4)$$

Perturbation theory in λ gives the following expansion for the ground state energy

$$E_0(R) = -\frac{\pi c_{\text{eff}}}{6R} - \frac{2\pi}{R} \sum_{n=1}^{\infty} C_n (R^y \lambda)^n . \quad (3.5)$$

The terms in the series come from the perturbative corrections, with coefficients

$$C_n = \frac{(-1)^n}{n!} R^{2-ny} \int \langle \Phi(0) \prod_{j=1}^{n-1} \Phi(\xi_j) d^2 \xi_j \rangle_{0,c} , \quad (3.6)$$

where the integration is taken over the cylinder and the indices $0,c$ refer to the connected contribution to the Green function in the original conformal field theory on the cylinder. For unitary theories, C_1 is zero, while in the case of non-unitary models the following formula is obtained in [17],

$$C_1 = -(2\pi)^{1-y} C_{\Phi_0 \Phi \Phi_0} , \quad (3.7)$$

where $C_{\Phi_0 \Phi \Phi_0}$ is the operator product coefficient for the perturbing operator Φ and the operator with the lowest (negative) conformal weight Φ_0 .

From the TBA, the same expansion can be computed using the conjectured scattering amplitudes. If we introduce the scaling length $r = mR$, where m is taken to be the mass of the lightest particle in the model, then the TBA yields $E_0(R)$ as the series

$$\frac{E_0(R)}{m} = -\frac{\pi \tilde{c}}{6r} - Ar - \frac{2\pi}{r} \sum_{n=1}^{\infty} a_n r^{yn} . \quad (3.8)$$

The first two terms, \tilde{c} and A , can be calculated analytically from the TBA whereas the a_n have to be calculated numerically. The agreement of the two series expansions (3.5) and (3.8) leads to the following relations:

$$c_{\text{eff}} = \tilde{c}, \quad a_n = C_n (m^y \lambda)^n . \quad (3.9)$$

The linear term in (3.8), which is absent in (3.5), is interpreted as the bulk energy term in finite volume.

The second relation in (3.9) yields a mass gap formula in the special case $n = 1$,

$$\lambda = \frac{a_1}{C_1} m^y . \quad (3.10)$$

The first nontrivial check is provided by the reproduction of the correct effective central charge, while the bulk energy constant and the mass-gap can be measured using TCSA. The $n > 1$ equalities could be used to perform further consistency checks, but we shall not do this.

Now we give the TBA results for the models $\mathcal{M}_{3,10}$ and $\mathcal{M}_{3,14}$ by relating them to known results.

3.1 TBA for $\mathcal{M}_{3,10} + \Phi_{(1,5)}$

It was pointed out in [18] that the TBA calculation for the scaling Yang-Lee model (which is $\mathcal{M}_{2,5} + \Phi_{(1,2)}$) is identical to the TBA calculation for the scaling Potts model $\mathcal{M}_{5,6} + \Phi_{(2,1)}$ for the following reason: The scaling Yang-Lee model contains one scalar particle B with the S -matrix

$$S_{BB} = \left(\frac{1}{3} \right) \left(\frac{2}{3} \right) , \quad (3.11)$$

while the spectrum of the scaling Potts model consists of a conjugate particle pair A and \bar{A} with the S -matrix

$$S_{AA} = \left(\frac{2}{3}\right) , \quad S_{A\bar{A}} = \left(\frac{1}{3}\right) . \quad (3.12)$$

The two S -matrices satisfy

$$S_{AA}S_{A\bar{A}} = S_{BB} . \quad (3.13)$$

As shown in [18], this means that the TBA equation in the charge symmetric Gibbs state for the scaling Potts model is equivalent to two copies of the scaling Yang-Lee model.

However, the model $\mathcal{M}_{3,10} + \Phi_{(1,5)}$ is explicitly equal to two copies of the Yang-Lee model, and therefore the ground state energy computed from the TBA of the model $\mathcal{M}_{5,6} + \Phi_{(2,1)}$ is identical to that of the model $\mathcal{M}_{3,10} + \Phi_{(1,5)}$.

3.2 TBA for $\mathcal{M}_{3,14} + \Phi_{(1,5)}$

The model $\mathcal{M}_{3,14} + \Phi_{(1,5)}$ is related to $\mathcal{M}_{6,7} + \Phi_{(1,2)}$. The S -matrix of the latter is the so-called minimal e_6 S -matrix, derived in [19]. If we denote the S -matrix of the i and j particles in the non-unitary model by $S_{ij}^{(3,14)}$, while that of the corresponding particles in the unitary case by $S_{ij}^{(6,7)}$, then we have the following interesting relation,

$$S_{ij}^{(3,14)} S_{i\bar{j}}^{(3,14)} = S_{ij}^{(6,7)} S_{i\bar{j}}^{(6,7)} , \quad (3.14)$$

even for the case when $S_{ij}^{3,14}$ is not a pure phase,

$$S_{AB} S_{A\bar{B}} = S_{lh} S_{l\bar{h}} . \quad (3.15)$$

This implies that the ground state energy calculated using the charge symmetric Gibbs state for TBA is the same in the two models.

The TBA for $\mathcal{M}_{6,7} + \Phi_{(1,2)}$ has been analysed in [17], where they find

$$\tilde{c} = 6/7 , \quad A^{TBA} = \frac{1}{6 + 2\sqrt{3}} , \quad a_1 = 0.0027765 \dots , \quad (3.16)$$

For $\mathcal{M}_{3,14}$, $c = -114/7$ and the operator Φ_0 with the minimal dimension is $\Phi_{(1,5)}$ with $h_{min} = -5/7$, so that $c_{\text{eff}} = 6/7$ as predicted. Using $C_{\Phi\Phi_0\Phi} = -20.3634 \dots$, the mass-gap m is given by

$$\lambda = \alpha m^{24/7} , \quad \alpha = -0.011833 \dots . \quad (3.17)$$

3.3 TBA for $\mathcal{M}_{3,16} + \Phi_{(1,5)}$

The arguments used in the two previous subsections to show that the TBA analysis of the two related models are equal are not valid here since the non-unitary model contains more particles in its spectrum than the corresponding unitary one. Therefore one does not expect the TBA prediction derived from the Ising model results to hold in this case.

For example, consider the effective central charge: In the $\mathcal{M}_{3,16} + \Phi_{(1,5)}$ model we have $c = -161/8$, $h_{min} = -7/8$ (corresponding to the operator $\Phi_{(1,5)}$) and so $c_{\text{eff}} = 7/8$. The related model is $\mathcal{M}_{3,4} + \Phi_{(1,2)}$ with $c_{\text{eff}} = 1/2$, and which has 8 particles with the minimal e_8 S -matrix, for which $\tilde{c} = 1/2$ as required.

In the e_8 TBA [17], the mass gap is expressed in terms of the lowest lying breather. If we try to use the e_8 TBA expansion for $\mathcal{M}_{3,16}$ given in [17], we get

$$a_1 = -0.001236 \dots, \quad C_{\Phi\Phi_0\Phi} = 51.85 \dots, \quad (3.18)$$

and therefore the predicted mass gap is

$$\lambda = \alpha m_{B_1}^{15/4}, \quad \alpha = -.003735 \dots \quad (3.19)$$

The bulk energy constant can be obtained from [17] as follows:

$$A^{TBA} = \frac{1}{4 \left(\sin \frac{\pi}{15} + \sin \frac{\pi}{3} + \sin \frac{2\pi}{5} + \sin \frac{3\pi}{5} + \sin \frac{2\pi}{3} + \sin \frac{14\pi}{15} \right)} = .06173 \dots \quad (3.20)$$

We give these simply to show that they do not agree with the numerical results of the TCSCA in section 4, as indeed they should not.

3.4 BA predictions from the S -matrix

The TBA system is a set of complicated non-linear integral equations for the ground-state energy $E_0(R)$, and even more complicated equations for higher energy levels [8]. There is a much simpler set of equations which give the dominant behaviour of two-particle excitations, which comes simply from the usual Bethe Ansatz for a two-particle wave function as follows.

On a circle the momenta of two-particle states are quantised. If we consider the zero momentum sector with particles i and j of masses m_i , m_j , momenta p , $-p$, and total energy $E(R)$, then we have

$$p = m_i \sinh(\theta_1) = -m_j \sinh(\theta_2), \quad E(R) - E_0(R) = \sqrt{p^2 + m_i^2} + \sqrt{p^2 + m_j^2}, \quad (3.21)$$

where $E_0(R)$ is the ground state energy. If the particles have purely elastic scattering with S -matrix $S_{ij}(\theta)$ then the quantisation condition is

$$\exp(ipR) S_{ij}(\theta_1 - \theta_2) = \pm 1, \quad (3.22)$$

with a -1 for i and j both fermions and 1 otherwise. Consequently it is possible to extract the S -matrix, or more properly the phase shift $\delta(\theta) = -i \log(S(\theta))$, from the function $E(R) - E_0(R)$. If there are two or more particles of the same mass with non-diagonal scattering then the quantisation condition is (3.22) with $S_{ij}(\theta)$ replaced by the eigenvalues of the two-particle S -matrix. We are able to identify several such two-particle lines in our numerical results and we can compare our conjectured S -matrices with those of the related models.

However, there is an ambiguity in our measurements of $S(\theta)$ which we should mention. The particle statistics are determined by the sign ± 1 in equation (3.22); there is also the

notion of ‘type’ which enters the TBA and which is given by $(\pm 1)S(0)$. The folklore is that all particles in theories such as ours are of fermionic type, that is $(\pm 1)S(0) = -1$, and this is the convention we have used in our TBA and BA calculations. However, the TBA and BA predictions are unaltered by $S(\theta) \rightarrow -S(\theta)$ and fermion \leftrightarrow boson, and this ambiguity can only be fixed by requiring the correct signs in the S -matrix bootstrap.

3.5 Extension of BA predictions from the S -matrix.

There are two situations in which we would like to consider extending equations (3.21) and (3.22) to complex rapidities.

Firstly, if the S -matrix is unitary, i.e. a pure phase for real rapidity, then we can reasonably expect $E(R)$ to be real. This requires that either θ_i are real and p is real, or θ_i is purely imaginary and p is purely imaginary. In the second case, the energy will be less than the sum of the particle masses, and so cannot correspond to an excited state of these two particles. However, if one considers the full TBA equations for the single particle state k , then it is quite possible that for large R that they are dominated by this BA equation, if k appears as a bound state pole in S_{ij} . Although this may give the leading corrections for large R , as is the case for the Yang-Lee model¹ to the $\mathcal{M}_{2,5} + \Phi_{(1,2)}$, typically this will not be a good approximation for small R as other corrections become dominant. However, we shall see that this need not be the case.

Secondly, if the S -matrix is not a pure phase for real rapidity difference, then requiring $E(R)$ real will lead to no solution of (3.22). However, in this case, we can instead demand that R is real, which is certainly a requirement, and which will typically lead to complex values of θ_i and $E(R)$. We have the first such occurrence in $\mathcal{M}_{3,14} + \Phi_{(1,5)}$, where S_{AB} and $S_{A\bar{B}}$ are not pure phases. They are exchanged under crossing, and as a result, the values of $E(R)$ which we find by applying the BA to these two S -matrices are conjugate. The BA is derived under the assumption that the particles are well separated and with real momenta. which is certainly not the case here; however it is remarkable that this simple rule does indeed describe the spectrum very well, and we compare these values with the TCSA data in subsection 4.1.

4 Numerical analysis in the Truncated Conformal Space Approach

The Truncated Conformal Space Approach (TCSA) consists of calculating the matrix elements of (4.1) exactly for a finite dimensional subspace \mathcal{H} of the full Hilbert space, and then numerically diagonalising the resulting matrix, as explained in [20]. Following the discussion in [20], the spectrum of the truncated Hamiltonian h_{TCSA} exhibits three distinct scaling regimes; r small in which the UV conformal field theory dominates, r large in which the truncation effects dominate and an intermediate region in which one hopes that the spectrum will approximate that of the massive theory on a circle. The TCSA method is fully described

¹One can check that the leading correction to the single-particle mass found in [9] is exactly given by the BA equation at imaginary rapidity

in e.g. [20], but we recall the main elements here. The full Hamiltonian is:

$$h(r) = \frac{2\pi}{r} \left(H_0 + \alpha \frac{r^y}{(2\pi)^{y-1}} H_1 \right) , \quad (4.1)$$

in units in which $m = 1$ and where

$$H_0 = L_0 + \bar{L}_0 - \frac{c}{12} . \quad (4.2)$$

The matrix elements of the perturbing Hamiltonian can be evaluated using the formula

$$\langle \Psi_f | V | \Psi_i \rangle = \frac{2\pi}{R} \lambda \frac{R^y}{(2\pi)^{y-1}} \langle \Psi_f | \Phi(1) | \Psi_i \rangle_{\text{plane}} , \quad (4.3)$$

where the matrix element of the perturbing field is taken on the conformal plane and evaluated using standard conformal field theory techniques. From this the operator H_1 can be identified with the perturbing field $\Phi(1)$ itself.

There are some symmetries which make the calculation of h_{TCSA} simpler: firstly, since the Hamiltonian is Lorentz invariant, we can restrict attention to the spin-zero sector of the full Hilbert space. Secondly, in our models the perturbing operator $\Phi(1) = \Phi_{(1,5)}(1)$ preserves distinct sectors of the full Hilbert space, and as a result we may diagonalise h_{TCSA} on these sectors separately. We can also choose to normalise the highest weight states as

$$\langle \Phi_j | \Phi_i \rangle = \pm \delta_{ij} , \quad (4.4)$$

where the signs are chosen to ensure the reality of the three-point couplings, which are themselves given in appendix A.

As discussed in section 3, the following numerical results can be extracted from the TCSA: (1) The ground-state energy: in the scaling regime the ground state energy should be approximately $E_0(r)/m \simeq Ar$, and one can measure A . (2) The mass-gap can be found and compared with the TBA predictions. In practice, we adopt the value of α from the TBA and then we expect that the lightest particle has mass 1. (3) Some of the two-particle lines can be identified, and for those, the two-particle energies can be compared with the Bethe Ansatz results derived from the S -matrix. We now present the results of the Truncated Conformal Space calculations for the two models $\mathcal{M}_{3,14} + \Phi_{(1,5)}$ and $\mathcal{M}_{3,16} + \Phi_{(1,5)}$.

4.1 The TCSA results for $\mathcal{M}_{3,14} + \Phi_{(1,5)}$

The allowed values of h are $h_{1,i}$, $0 < i < 14$. It is possible to consider several different sectors in which $\Phi_{(1,5)}$ acts locally but which are not themselves necessarily mutually local. In particular, we can consider the three sectors ‘even’, ‘odd’ and ‘twisted’ for which the fields all have integer spin, with the following sets of $\{h_{1,i}, \bar{h}_{1,j}\} \equiv [i, j]$,

Sector	Field content	
even	$[1, 1], [3, 3], [5, 5], [7, 7], [9, 9], [11, 11], [13, 13]$	
odd	$[2, 2], [4, 4], [6, 6], [8, 8], [10, 10], [12, 12]$	
twisted	$[1, 13], [3, 11], [5, 9], [7, 7], [9, 5], [11, 3], [13, 1]$	

(4.5)

The coupling constants for first two sectors are those of the ‘A’ type modular invariant, which we give in appendix A, the even and twisted sectors correspond to the ‘D’ type modular invariant.

To establish that this model has the spectrum we have predicted, it is only necessary to consider the ‘even’ sector. We truncated the space by requiring $L_0 + \bar{L}_0 \leq 10$ which leaves 285 states. In figure 1 we give the real eigenvalues of $h(r)$ and pick out the first 13 lines; it is clear that scaling has set in for the first three eigenvalues by $r = 10$.

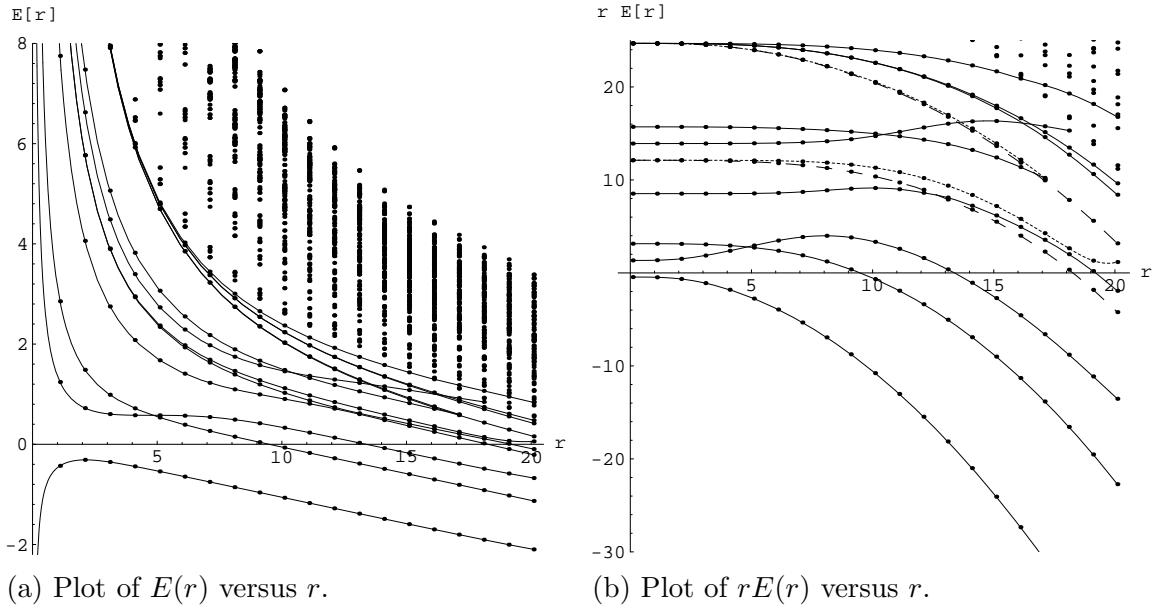


Figure 1: The first 13 eigenvalues of $h(r)$ in $\mathcal{M}_{3,14} + \Phi_{(1,5)}$.

Some of the lines appear to terminate – in fact two real eigenvalues collide and become a complex conjugate pair of eigenvalues; this is due to truncation effects as explained in [9].

From figure 1a it is possible to estimate the slope of the ground-state energy and the first three excitation energies in the scaling region (which we take to be $13 < r < 17$), and we give these in table 2 for truncation levels 8, $9\frac{1}{7}$ and 10 to show systematic errors together with estimates of the error of the measurements, together with the values predicted by the TBA and perturbation theory.

Level	$\dim \mathcal{H}$	A	m_1	m_2	m_3
8	136	-0.105 ± 0.05	0.994 ± 0.006	1.420 ± 0.006	1.94 ± 0.03
$9\frac{1}{7}$	280	-0.106 ± 0.02	1.01 ± 0.02	1.417 ± 0.002	1.95 ± 0.03
10	285	-0.105 ± 0.05	0.996 ± 0.004	1.415 ± 0.004	1.95 ± 0.03
Exact		$-0.1056\dots$	1	$1.414\dots$	$1.932\dots$

Table 2: Results for $\mathcal{M}_{3,14} + \Phi_{(1,5)}$

However, as we discuss further in section 5, the particles A and \bar{A} , and B , \bar{B} we used to diagonalise the S -matrix in section [1] are not invariant under Z_2 and consequently there is no direct connection between these particles and the excitations in the even sector; rather the particle of lowest mass in the even sector is a symmetric combination of A and \bar{A} ; similarly, the zero-momentum two-particle states of A and \bar{A} in the even sector are $|K(\theta)K(-\theta)\rangle$ and $|\tilde{K}(\theta)\tilde{K}(-\theta)\rangle$. However, it is still true that the eigenvalues of the S -matrix on these two particle states are the matrix elements $S_{AA}(\theta)$ and $S_{A\bar{A}}(\theta)$, and we have indicated the first two lines corresponding to the eigenvalues by dotted and dashed lines respectively.

From figure 1b we can extract the S -matrix element $S_{AA}(\theta)$, which we compare with the conjectured results in figure 2a; the data is from the *second* energy level which corresponds to these particles, i.e. the 10th eigenvalues of $h(r)$ for small r^2 . As is usual, there is a small discrepancy for small r between the Bethe-Ansatz predictions for $E(r)$ and the actual values due to finite-size effects, but otherwise the agreement is very good. In figure 2b we give the same data for $S_{CC}(\theta)$ which corresponds to the 13th eigenvalue of $h(r)$ for small r . The results for S_{BB} and $S_{B\bar{B}}$ are equally convincing.

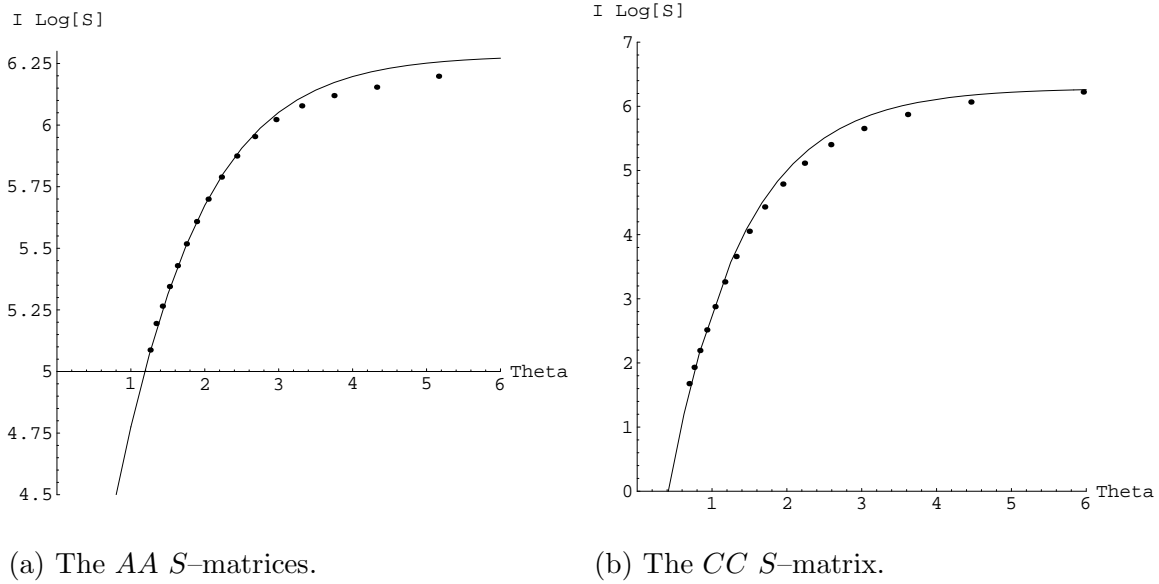


Figure 2: The S -matrices as extracted from the TCSA compared with theory in $\mathcal{M}_{3,14} + \Phi_{(1,5)}$.

The interpretation of the lowest energy solution of equation (3.22) for $S = S_{A\bar{A}}$ (the 5th eigenvalue for small r) is extremely interesting. Since $S_{AA}(0) = -1$, the particles A and \bar{A} are bosons, and so the Bethe Ansatz for the n th $A\bar{A}$ excitation is

$$r \sinh(\theta) + \delta_{A\bar{A}}(2\theta) = 2n\pi . \quad (4.6)$$

Since $\delta_{A\bar{A}}(\theta) \sim -\theta(3 + 2\sqrt{2})$ for small θ we see that equation (4.6) with $n = 0$ only has solutions for real θ for $r \leq 6 + 4\sqrt{3} = 12.92\dots$. However, for r large, we have identified this

² In figs. 2a and 2b, small r corresponds to large θ , and large r to small θ .

5th line as the single particle Z_2 even component of the higher kink doublet B, \bar{B} , and hence obtained the estimate of the mass $m_3 = 1.95 \dots$

We can reconcile these two interpretations when we notice that for $r > 6 + 4\sqrt{3}$ the solution to equation (4.6) corresponds to θ imaginary. As $r \rightarrow \infty$, we see that $\theta \rightarrow i\pi/6$, and there is a pole in $S_{A\bar{A}}(\theta)$ at $\theta = i\pi/6$ which corresponds to the particles B and \bar{B} . Thus we see direct evidence that the first excited kink (corresponding to the particles B and \bar{B}) appears as the direct channel pole of KK at $\theta = i\pi/6$.

As a consequence, the TCSA gives the value of the S -matrix $S_{A\bar{A}}(\theta)$ for all real θ and also for the imaginary values between 0 and $i\pi/6$. In figure 3 we plot the theoretical and numerical values of both $-\delta_{A\bar{A}}(\theta)$ versus θ and $\log(S(\theta))$ versus $-i\theta$. There is excellent agreement for all real values of θ , and also for imaginary values of θ up to $r \sim 17$ (corresponding to $\theta \sim 0.4i$) where scaling starts to break down and truncation effects take over³.

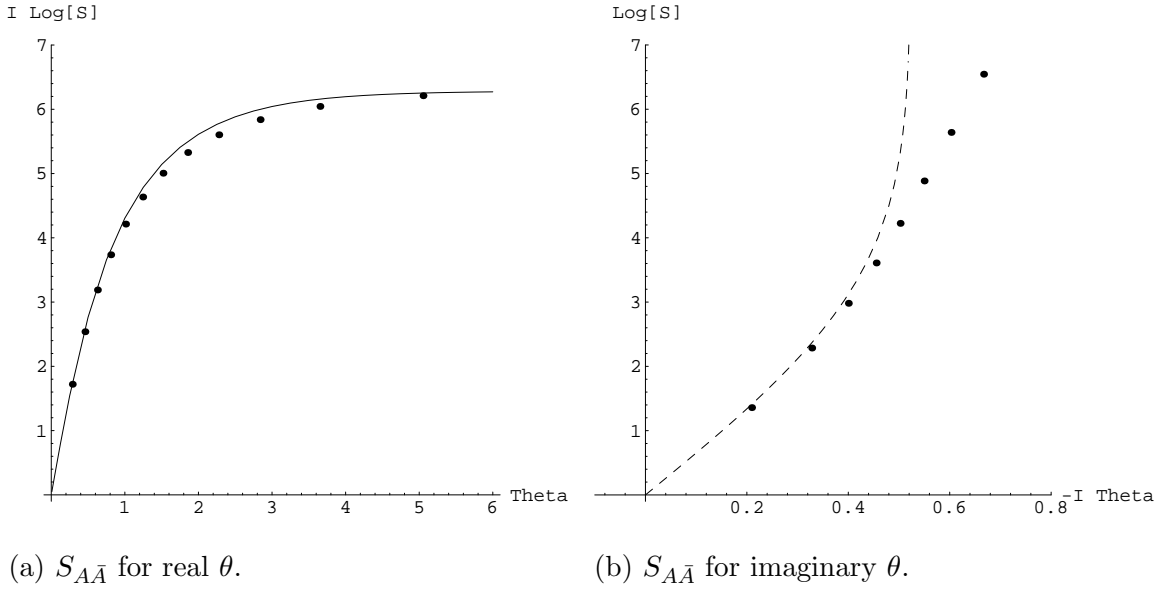


Figure 3: The S -matrices as extracted from the TCSA compared with theory in $\mathcal{M}_{3,14} + \Phi_{(1,5)}$.

However, figure 1 does not tell the whole story, as we have only plotted the real eigenvalues. There are also eigenvalues which are genuinely complex, and in figure 4 we show the real and imaginary parts of those eigenvalues λ amongst the first 60, for which $\Im m(\lambda) > 10^{-6}$. While some of these are simply the results of truncation error, there are also several series of lines which are not, and for these we have included the predictions of the ‘two-particle BA’ using the S -matrices S_{AB} and $S_{A\bar{B}}$ and as can be seen, there is excellent agreement.

³In figs. 3a and 3b, small r corresponds to large θ , and $r \rightarrow \infty$ to $\theta \rightarrow i\pi/6$

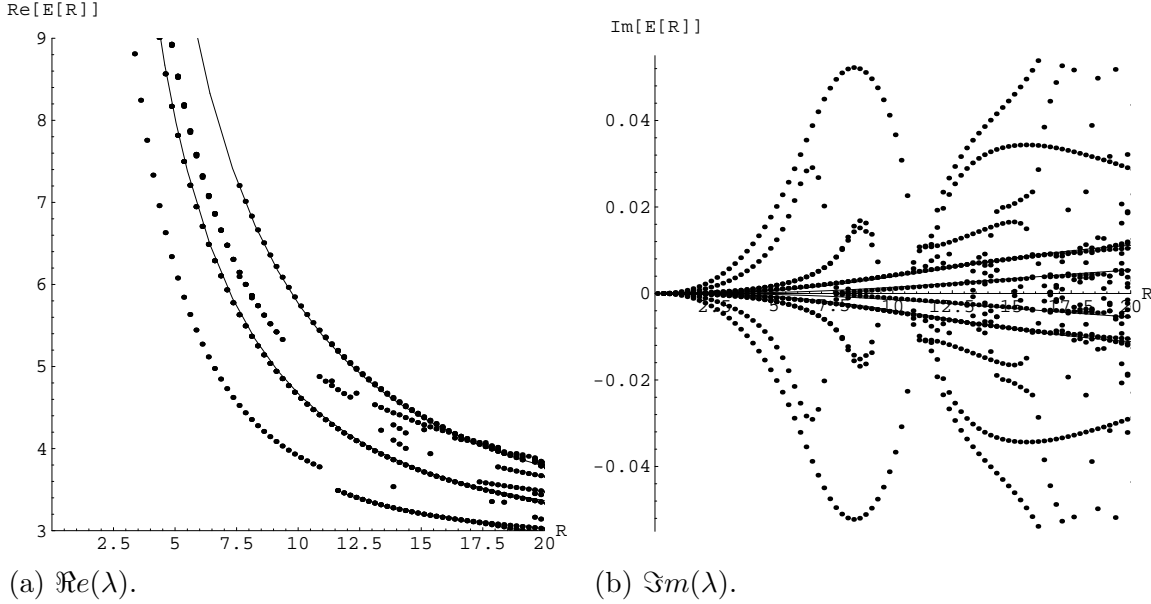


Figure 4: The complex eigenvalues $h(r)$ in $\mathcal{M}_{3,14} + \Phi_{(1,5)}$: data and BA predictions

4.2 The TCSA results for $\mathcal{M}_{3,16} + \Phi_{(1,5)}$

The allowed values of h are $h_{1,i}$, $0 < i < 16$. It is again possible to consider several different sectors in which $\Phi_{(1,5)}$ acts locally but which are not themselves necessarily mutually local. In particular, we can again define three sectors ‘even’, ‘odd’ and ‘twisted’ for which the fields all have integer spin, with the following sets of $\{h_{1,i}, \bar{h}_{1,j}\} \equiv [i, j]$,

Sector	Field content	
even	$[1, 1], [3, 3], [5, 5], [7, 7], [9, 9], [11, 11], [13, 13], [15, 15]$	(4.7)
odd	$[2, 2], [4, 4], [6, 6], [8, 8], [10, 10], [12, 12], [14, 14]$	
twisted	$[2, 14], [4, 12], [6, 10], [8, 8], [8, 8], [10, 6], [12, 4], [14, 2]$.	

The coupling constants for first two sectors are those of the ‘A’ type modular invariant, which we give in appendix A, the twisted sector corresponds to the ‘D’ type modular invariant.

To establish that this model has the spectrum we have predicted, it is again only necessary to consider the ‘even’ sector. We truncated the space by requiring $L_0 + \bar{L}_0 \leq 12$ which leaves 870 states. In figure 5 we give the real eigenvalues of $h(r)$ with the first 14 lines picked out. It is less clear that scaling has set in for the first five eigenvalues, and both the masses of the single particles and the S -matrices as extracted from the excited two-particle lines are in good agreement with theory. However, to achieve a reliable scaling regime it has been necessary to use a much larger truncated space for this model than the previous one; indeed when the space is truncated to 285 states the lines corresponding to the third particle are no longer present, and for that reason we leave the measurement of its mass blank in table 3. The severity of the truncation effects which still remain can also be seen in the large gaps in the 2nd, 3rd, 4th and 5th lines (when the eigenvalues become complex) and which would

presumably disappear when the truncation level is increased.

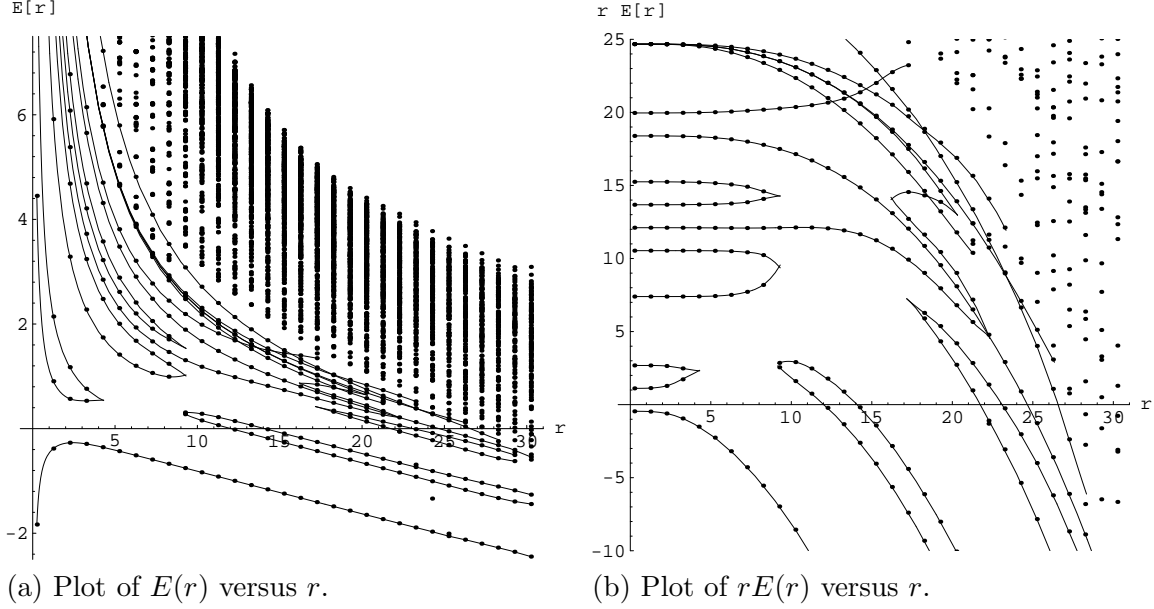


Figure 5: The first 14 eigenvalues of $h(r)$ in $\mathcal{M}_{3,16} + \Phi_{(1,5)}$.

From figure 5a it is possible to estimate the slope of the ground-state energy and the first three excitation energies in the scaling region (which we take to be $10 < r < 12$), and we give these in table 3 for the truncation levels $8\frac{3}{4}$, $9\frac{3}{4}$ and 12 to show systematic errors together with estimates of the error of the measurements.

Level	$\dim \mathcal{H}$	A	m_1	m_2/m_1	m_3/m_1
$8\frac{3}{4}$	285	-0.08 ± 0.02	1.01 ± 0.06	1.14 ± 0.04	—
$9\frac{3}{4}$	369	-0.081 ± 0.05	0.96 ± 0.03	1.175 ± 0.02	1.81 ± 0.02
12	870	-0.0815 ± 0.0008	0.975 ± 0.006	1.178 ± 0.02	1.83 ± 0.02
Exact				1.175...	1.827...

Table 3: Results for $\mathcal{M}_{3,16} + \Phi_{(1,5)}$

It is clear from the measurement of A that this is not given by the standard minimal e_8 TBA, as should be the case, and that the mass ratios are consistent with those of the S -matrices of [1].

From figure 5b we can extract the eigenvalues of the S -matrices of the fundamental kinks, which we compare with the conjectured results in figure 6. The data points come from the 9th, 11th and 23rd eigenvalues of $h(r)$ (for r small) with the continuous lines being $S_1(\theta)$, $S_2(\theta)$, and the dashed lines $S_3(\theta)$ and $S_4(\theta)$.

We can also identify the first two-breather line, which is the 14th eigenvalue of $h(r)$ for small R , and we compare the S -matrix extracted from these numerical results with the theoretical

result of

$$m_{B_1} = 2m \sin \frac{\pi}{15} = 1.175 \dots m, S_{B_1 B_1} = \left(\frac{2}{30}\right) \left(\frac{10}{30}\right) \left(\frac{12}{30}\right) \left(\frac{18}{30}\right) \left(\frac{20}{30}\right) \left(\frac{28}{30}\right). \quad (4.8)$$

in figure 6. In this case we have $S_{B_1 B_1}(0) = 1$ so that we take B_1 fermionic for the Bethe Ansatz, in accord with the folklore that bosonic and fermionic particles have $S(0) = -1$ and 1 respectively; in any case there is agreement between theory and experiment for this choice.

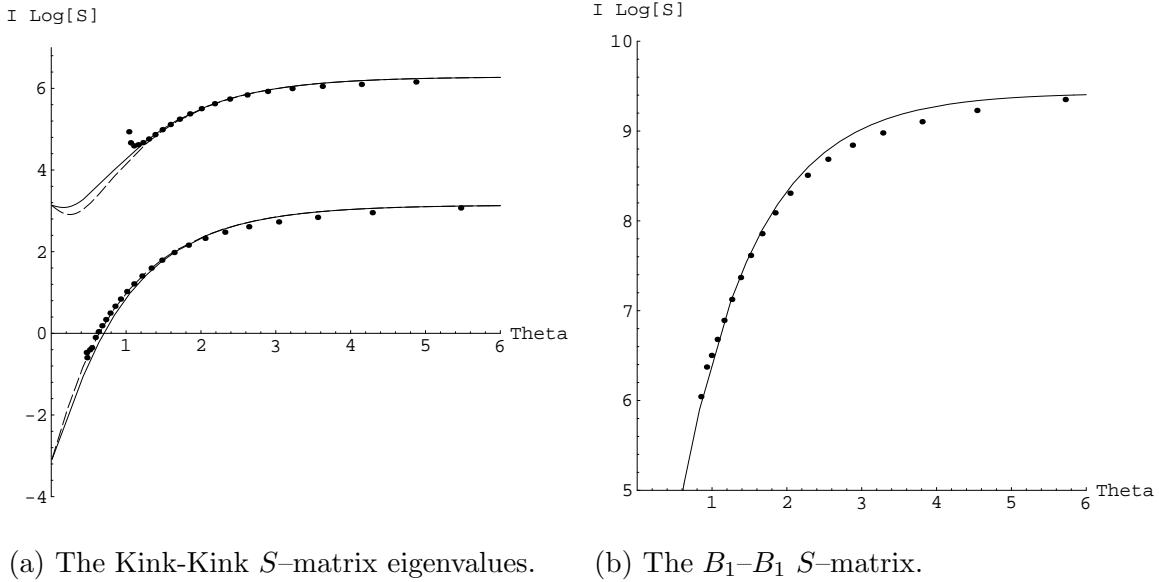


Figure 6: The S -matrices as extracted from the TCSA compared with theory in $\mathcal{M}_{3,16} + \Phi_{(1,5)}$

5 Z_2 symmetry and sectors

Minimal models of CFT are known to exhibit Z_2 symmetry, which has been used above to classify the sectors of the Hilbert-space. These sectors are preserved under $\Phi_{(1,5)}$ perturbation and therefore it is natural to expect a corresponding symmetry of the massive field theory.

There is a natural Z_2 action on the particles in the unrestricted ZMS model, completely analogous to the classification of the irreducible representations of the classical group $SU(2)$ by its centre Z_2 , with integer spin representations corresponding to the trivial (even) class and half-integer spin representations to the nontrivial (odd) class. Although the $\mathcal{U}_q(sl(2))$ action does not survive RSOS restriction, the Z_2 action does, and as a consequence, the kinks K are even and \bar{K} odd. Since the Z_2 symmetry is a remnant of the full quantum group symmetry, it commutes with the S -matrix, and it is possible to simultaneously diagonalise both the S -matrix and Z_2 on the two-particle states. However, it is *not* possible to do this on one-particle states. As a result, the one-particle states which diagonalise the S -matrix (denoted A , \bar{A} in case of $\mathcal{M}_{3,14}$ and $\mathcal{M}_{3,16}$ and A , B and the case of $\mathcal{M}_{3,10}$) are *not* Z_2 eigenstates; rather, they are exchanged with each other under Z_2 . Therefore the natural description of

the content of each sector is in terms of the states K and \tilde{K} , which are self-conjugate and Z_2 -even or odd, respectively.

We now examine the content of the three different sectors (even, odd and twisted) of the simplest model $\mathcal{M}_{3,10} + \Phi_{(1,5)}$, as an example. Exactly as for the models $\mathcal{M}_{3,14}$ and $\mathcal{M}_{3,16}$ the values of $\{h_{1,i}, \bar{h}_{1,j}\} \equiv [i, j]$ of $\mathcal{M}_{3,10}$ fall into even, odd and twisted as follows:

Sector	Field content	
even	$[1, 1], [3, 3], [5, 5], [7, 7], [9, 9],$	(5.1)
odd	$[2, 2], [4, 4], [6, 6], [8, 8],$	
twisted	$[1, 9], [3, 7], [5, 5], [7, 3], [9, 1].$	

The direct sum of the odd and twisted sectors correspond to the product $(\mathcal{M}_{2,5} + \Phi_{(1,2)})^{\otimes 2}$, and give the D type modular invariant in the conformal limit, The direct sum of the even and odd sectors gives the A -type modular invariant partition function. Direct TCSA investigation of the odd sector gives very similar pictures to those of the even sector in the scaling regime: both sectors have a ground state, a one-particle line and a very similar structure of two-particle lines. The degeneracy of the ground state energies is broken by an amount which is exponentially small for large r . The picture is very similar for $\mathcal{M}_{3,14} + \Phi_{(1,5)}$.

For the model $\mathcal{M}_{(3,16)}$, the four eigenvalues of the kink–kink S -matrix are different, so that we now expect that the two-particle lines in the even and odd sectors are given by different Bethe Ansatz results. In the even sector, we expect the two-particle lines to be linear combinations of the states $|K(\theta)K(-\theta)\rangle$ and $|\tilde{K}(\theta)\tilde{K}(-\theta)\rangle$, whereas those in the even sector to be linear combinations of $|K(\theta)\tilde{K}(-\theta)\rangle$ and $|\tilde{K}(\theta)K(-\theta)\rangle$. The identification above is consistent with the fact that the eigenvalues on the Z_2 even sector, (S_3 and S_4 , the dashed lines) give a better approximation to the phase-shift extracted from the even sector, than those on the Z_2 odd sector (S_1 and S_2 , the solid lines).

This picture can be formulated generically as follows: states corresponding to RSOS sequences $\{j_n, \dots, j_2, j_1\}$ which start and end with an even vacuum state (i.e. j_1 and j_n are integer-spin representations, which in our three cases means the state $|0\rangle$) originate from the even sector of the underlying CFT, and the ones starting with an odd vacuum and ending with an odd vacuum derive from the odd sector.

6 Character Identities

As has been shown extensively, there are interesting sum forms for Virasoro characters which may be related to the particle structure of perturbed minimal models (see e.g. [21] for the cases related to the minimal e_8 S -matrix.). Since the breather sectors of our related models are the same, and the characters of the Virasoro representations in these models may be given in terms of sum formulae, we might expect to find relations between the Virasoro characters in the two models. This is indeed the case, as we show now. The character of the irreducible representation $L_{c,h}$ of the Virasoro algebra of central charge c with highest weight h is defined as

$$\chi_{c,h}(q) = \text{Tr}_{L_{c,h}} \left(q^{L_0 - c/24} \right) .$$

For pairs $\mathcal{M}_{r,s}$ and $\mathcal{M}_{r',s'}$ of type II ($r' = r/2, s' = 2s$) we find that

$$\begin{aligned}\chi_{c_{r,s},h_{r/2-2m,n}} - \chi_{c_{r,s},h_{r/2-2m,s-n}} &= \chi_{c_{r',s'},h_{m,s'/2-2n}} - \chi_{c_{r',s'},h_{r'-m,s'/2-2n}}, \\ \chi_{c_{r,s},h_{2m,n}} &= \chi_{c_{r',s'},h_{m,2n}}.\end{aligned}\tag{6.1}$$

Note that these identities do not involve all the characters of either model. For the pair $\mathcal{M}_{3,10}$ and $\mathcal{M}_{5,6}$ these relations give

$$\begin{aligned}\chi_{c_{3,10},h_{1,1}} - \chi_{c_{3,10},h_{1,9}} &= \chi_{c_{5,6},h_{2,1}} - \chi_{c_{5,6},h_{3,1}}, \\ \chi_{c_{3,10},h_{1,3}} - \chi_{c_{3,10},h_{1,7}} &= \chi_{c_{5,6},h_{1,1}} - \chi_{c_{5,6},h_{4,1}}, \\ \chi_{c_{3,10},h_{1,2}} &= \chi_{c_{5,6},h_{1,2}}, \\ \chi_{c_{3,10},h_{1,4}} &= \chi_{c_{5,6},h_{2,2}}, \\ \chi_{c_{3,10},h_{1,6}} &= \chi_{c_{5,6},h_{3,2}}, \\ \chi_{c_{3,10},h_{1,8}} &= \chi_{c_{5,6},h_{4,2}}.\end{aligned}\tag{6.2}$$

For the pairs $\mathcal{M}_{r,s}$ and $\mathcal{M}_{r,4s}$ of type III we find that

$$\chi_{c_{r,s},h_{m,n}} = \begin{cases} \chi_{c_{r,4s},h_{(r-m)/2,2(s-n)}} - \chi_{c_{r,4s},h_{(r-m)/2,2(s+n)}} & m \text{ odd} \\ \chi_{c_{r,4s},h_{m/2,2n}} - \chi_{c_{r,4s},h_{m/2,4s-2n}} & m \text{ even} \end{cases}.\tag{6.3}$$

In the case of $\mathcal{M}_{3,16}$ and $\mathcal{M}_{3,4}$ this yields

$$\begin{aligned}\chi_{c_{3,4},h_{1,1}} &= \chi_{c_{3,16},h_{1,6}} - \chi_{c_{3,16},h_{1,10}}, \\ \chi_{c_{3,4},h_{1,2}} &= \chi_{c_{3,16},h_{1,4}} - \chi_{c_{3,16},h_{1,12}}, \\ \chi_{c_{3,4},h_{1,3}} &= \chi_{c_{3,16},h_{1,2}} - \chi_{c_{3,16},h_{1,14}}.\end{aligned}\tag{6.4}$$

Analogous relations for type I pairs are obtained by swapping $r \leftrightarrow s', s \leftrightarrow r'$ and $m \leftrightarrow n$. All these relations may be proven easily using the formulae in [22]. Whether the identities in equations (6.1), (6.3) have any real interpretation in terms of the particle structures of PCFTs is an interesting question to which we hope to have an answer shortly, although it is interesting to note that in each case we have investigated when the difference of two characters appears, it still has an expansion in q with positive coefficients.

7 Conclusions

The above results make it possible to clarify the connection between $\Phi_{(1,2)}$ and $\Phi_{(1,5)}$ perturbations of minimal models. We have shown that they can be related through the ZMS Lagrangian and that the relation falls into one of the three possible classes of type I, II or III, according to the relative magnitude of the effective central charges. Using the S-matrices obtained via RSOS restriction in [1], we tested them against the predictions of the TCSCA and TBA method. We have confirmed the picture of $\Phi_{(1,5)}$ perturbations based on the RSOS restriction and have shown that the two related theories are different reductions of the same underlying affine Toda field theory. The existence of such different restrictions relies upon the existence of non-equivalent $sl(2)$ subalgebras of the affine Kac-Moody algebra $a_2^{(2)}$ underlying the ZMS model. It appears reasonable to expect that such relations should exist between perturbations of W -minimal models corresponding to other imaginary coupling affine Toda field theories.

The results show that in the type II case (models $\mathcal{M}_{(3,10)}$ and $\mathcal{M}_{(3,14)}$) we end up with the same mass spectra, although different scattering theories, whereas in the cases of type I (of which the example of $\mathcal{M}_{2,9}$ was treated in [6, 7]) and type III (of which $\mathcal{M}_{3,16}$ is an example) even the mass spectra are different just as expected. In the two type II cases the TBA equations turned out to be the same as those of the corresponding $\Phi_{(1,2)}$ -perturbed conformal field theories.

We have also shown that the strange features of the S -matrices for $\mathcal{M}_{3,14} + \Phi_{(1,5)}$ found in [1] are in fact correct. This model does indeed break unitarity much more severely than the $\Phi_{(1,2)}$ and $\Phi_{(1,3)}$ perturbations considered before. The spectrum is not entirely real, and it is well described by the S -matrices no longer being pure phases. We believe that this is a generic feature of RSOS restrictions, and that the previous situation with a real spectrum is in fact rather exceptional. Indeed, looking at the S -matrices found for $a_2^{(1)}$ Toda theory in [24] we see that the S -matrices for soliton-breather scattering (which should not be altered by RSOS reduction) are also not pure phases in general.

We have also shown that in one case, the Bethe Ansatz equations give a good approximation for a single-particle state energy by continuation to imaginary rapidity. While it is easy to check that in general this is not the case, it is true that the leading corrections to the particle mass in $\mathcal{M}_{2,5} + \Phi_{(1,2)}$ are also given by the BA equations, and that this can be seen by examining the large R behaviour of the full TBA equations describing this energy [8]. We believe that in our case, the full TBA equations are simply dominated by the BA equations for all values of R , rather than simply for large R .

Unfortunately, due to our lack of knowledge about the conformal field theory three-point coupling constants for D -type modular invariants of non-unitary models, we have not been able to extract precise TCSA predictions for the twisted sector, although preliminary results (using some guesses for the structure constants) have turned out to be in good agreement with our expectations based on the Z_2 picture. It seems worthwhile to tackle the problem of the coupling constants in the non-unitary minimal models and in this way to facilitate the further investigation of their structure as well.

Since the RSOS restriction does not modify the breather sector of the affine Toda field theory, one can expect that this sector is identical in corresponding $\Phi_{(1,2)}$ and $\Phi_{(1,5)}$ perturbations irrespective of the type I, II or III nature of the pair. This leads to the existence of ‘strange’ character identities between characters of different minimal models. In addition to that, in the type II case we expect relations between the states corresponding to the solitonic sectors as well. Some character identities of this type have been described, although it is not clear whether they originate from some connections between different PCFTs in the case of general minimal model pairs.

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A Coupling constants in non-unitary conformal field theories

The coupling constants for fields of type $\Phi_{(1,a)}$ are given in [12] in an asymmetric fashion: they have two sets of fields $\varphi_a, \bar{\varphi}_a$ which for our purposes are just (different) multiples of $\Phi_{(1,a)}$. Using the results of [12] we arrive at

$$\langle \varphi_a | \varphi_b(1) | \varphi_c \rangle = \tilde{C}_{abc} , \quad \langle \varphi_a | \varphi_b \rangle = \tilde{C}_{ab1}$$

where the coupling constants are

$$\tilde{C}_{abc} = \frac{\xi(\frac{b+c-a-1}{2})}{\xi(a-1)} \frac{\xi(\frac{a+c-b-1}{2})}{\xi(b-1)} \frac{\xi(\frac{a+b-c-1}{2})}{\xi(c-1)} \frac{\eta(\frac{a+b+c-1}{2})}{\eta(1)} , \quad (\text{A.1})$$

where

$$\xi(a) = \prod_{j=1}^a \gamma(\rho j) , \quad \eta(a) = \prod_{j=1}^a \gamma(\rho j - 1) , \quad \gamma(x) = \Gamma(x)/\Gamma(1-x) , \quad \rho = r/s .$$

From the unnormalised coupling constants \tilde{C}_{abc} it is then easy to choose normalised fields $\Phi_{(1,a)} = \frac{1}{\alpha_a} \varphi_a$ such that $\langle \Phi_{(1,a)} | \Phi_{(1,b)}(1) | \Phi_{(1,c)} \rangle$ is real and $\langle \Phi_{(1,a)} | \Phi_{(1,a)} \rangle = \pm 1$.

We do not give the coupling constants for the D type modular invariant involving the fields in the twisted sector as these are not to be found in [12]. Petkova and Zuber [23] give these for the unitary minimal models, and it would be interesting to extend their results to all minimal models, but we do not do that here.

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